Testing the APT with the Maximum Sharpe Ratio of Extracted Factors - Erratum

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August, 2009

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1. Error

In the Arbitrage Pricing Theory (APT) literature, the return-generating processes are usually stated in two ways. The first is to write

$$r_t = a + Bf_t + \varepsilon_t, \tag{1}$$

where r_t is the n-vector of returns in excess of the riskfree rate over period t, f_t is the k-vector of systematic factors over t, $E\varepsilon_t = 0$ and $Ef_t\varepsilon_t' = 0$. If f_t is chosen as excess returns on factor-mimicking portfolio, then $Ef_t \equiv \mu_f$ is the factor premium, $\mu \equiv Er_t = a + B\mu_f$, and a is vector of pricing errors associated with f_t relative to the exact version of the APT: $\mu = B\mu_f$. It is important to note that $B'a = 0$ as assumed in the APT. The second way is to write

$$r_t = \mu + B\tilde{f}_t + \varepsilon_t, \tag{2}$$

where $E\tilde{f}_t = 0$, $E\varepsilon_t = 0$ and $E\tilde{f}_t\varepsilon_t' = 0$. The two expressions are equivalent with $\tilde{f}_t = f_t - \mu_f$ and $\mu = a + B\mu_f$.

Let $S_r = Er_tr_t'$, $S_f = Ef_tf_t'$, $S_{\tilde{f}} = E\tilde{f}_t\tilde{f}_t'$ and $\Sigma_\varepsilon = E\varepsilon_t\varepsilon_t'$ be the second moment matrices of the corresponding variables. Then from the two expressions of returns,

$$\begin{aligned} S_r &= aa' + BS_fB' + \Sigma_\varepsilon + a\mu_f'B' + B\mu_f a' & (3)\\ &= \mu\mu' + BS_{\tilde{f}}B' + \Sigma_\varepsilon. & (4)\end{aligned}$$

The published paper uses (1), However, Equation (2) in the paper mixes up the two expressions of S_r above by writing $S_r = aa' + BS_fB' + \Sigma_\varepsilon$, which is wrong for any finite n. The correct equation is (3) here.

The error does not affect the validity of the rest of the paper, however. Only the proof of Proposition 1 should be revised. Rewrite $S_r = aa' + B_gB_g' + \Sigma_\varepsilon + a\mu_g'B_g' + B_g\mu_g a'$ where $B_g = BS_f^{1/2}$ and $\mu_g = S_f^{-1/2}\mu_f$. Since by definition, Σ_ε has bounded eigenvalues, the number of unbounded eigenvalues of S_r is the same as the number of unbounded eigenvalues of

$S = aa' + B_g B_g' + a\mu_g' B_g' + B_g \mu_g a'$. It can be shown that the $k+1$ positive eigenvalues of S satisfy the following equation

$$\alpha + \sum_{j=1}^{k} \alpha \mu_j^2 \frac{\beta_j}{\lambda - \beta_j} = \lambda, \tag{5}$$

where $\alpha = a'a$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k > 0$ are the positive eigenvalues of BB', and $\mu_g = (\mu_1, \cdots, \mu_k)'$. If $\beta_1, \cdots, \beta_k, \alpha$ all tend to infinity, it's easy to see that any solution to (5) tend to infinity. If $\beta_1, \cdots, \beta_k$ tend to infinity, but $\alpha \to \bar{\alpha} < \infty$ (or remains bounded), then it can be verified that in the limit, $\lambda/\beta_j \to 1$ for $j = 1, \cdots, k$ for the k largest eigenvalues and $\lambda \to \bar{\alpha}(1 - \mu_g'\mu_g) > 0$, which is finite, (or remain bounded) for the smallest positive eigenvalue.

2. Typo

There is a typo in the proof of Proposition 2 (iii). It is a typo made by the publisher, which the author missed in the galley proof. In the published paper,

> ... It follows that, in the limit when n goes to infinity, the maximum squared Sharpe ratio is
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> $$\begin{aligned} s &= \lim_{n\to\infty} \mu_r' \Sigma_r^{-1} \mu_r = \lim_{n\to\infty} (a + B_g\mu_g)'(B_g \Sigma_g B_g' + \Sigma_\varepsilon)^{-1}(a + B_g\mu_g) \\ &= \lim_{n\to\infty} (B_g\mu_g)'(B_g\Sigma_g B_g')^{+}(B_g\mu_g) = \mu_g'\Sigma_g^{-1}\mu_g = \mu_g'(I_k - \mu_g\mu_g')^{-1}\mu_g \\ &= \mu_g'[I_k + \mu_g\mu_g'/(1 + \mu_g'\mu_g)]\mu_g = \gamma/(1-\gamma), \end{aligned}$$
>
> where ...

The typo occurs in the last line of the formulas, which should be

$$= \mu_g'[I_k + \mu_g\mu_g'/(1 - \mu_g'\mu_g)]\mu_g = \gamma/(1-\gamma).$$